Functions of a Complex Variable

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Introduction to Complex Functions

Let z and w be complex variables, A complex valued function of a complex variable is denoted by w=f(z).

Examples:

- The function w=iz+3 is defined in the entire complex plane.
- The function $w = \frac{1}{z^2+1}$ is defined at all points of the complex plane except at z= $\pm$ i.
- $w = |z|$ is defined in the entire complex plane and this is a real-valued function of a complex variable.

Example: Polynomial and Rational Functions

- If $a_0, a_1, a_2, \ldots. a_n$ are complex constants, the function $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots. + a_n z^n$ is defined in the entire complex plane and is called a polynomial in z.

- If P(z) and Q(z) are polynomials of the portions $\frac{P(z)}{Q(z)}$ is called a rational function and it is defined for all z with $Q(z) \neq 0$.

General Form of Complex Functions.

If u(x,y) and v(x,y) are real-valued functions of two variables defined in a region of the complex plane, then f(z)=u(x,y)+iv(x,y) is a complex-valued function defined on that region.

Conversely,

Every complex function w=f(z) can be put in the form w=f(z)=u(x,y)+iv(x,y), where u and v are real-valued functions of the real variables x and y.

Example:

$$f(z) = z^2$$

Put $z = x + iy$

$$f(z) = z^2$$

$$= (x + iy)^2$$

$$= x^2 - y^2 + 2ixy$$

$$f(z) = x^2 - y^2 + i2xy$$

Here, $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

Thus a complex function w=f(z) can be viewed as a function of the complex variable z or as a function of two real variables x and y.

LIMITS

DEFINITION:

A function $w = f(z)$ is said to have the limit l as z tends to z_0, if given $\xi > 0$ there exist a $\delta > 0$, such that $0 < |z - z_0| < \delta$ which implies $|f(z) - l| < \xi$.

That is $\underset{z \to z_0}{lt} f(z) = l.$

LEMMA

When the limit of a function $f(z)$ exists as z tends z_o ,then the limit has a unique value.

PROOF:

Suppose that $\underset{z \to z_0}{lt} f(z)$ has two values l_1 and l_2, then given $\xi > 0$ there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < |z - z_0| < \delta_1 \Rightarrow |f(z) - l_1| < \frac{\xi}{2} \text{ -------- (1)}$$

$$0 < |z - z_0| < \delta_2 \Rightarrow |f(z) - l_2| < \frac{\xi}{2} \text{ -------- (2)}$$

Now, Let $\delta = \min\{\delta_1, \delta_2\}$

Then ,if $0 < |z - z_0| < \delta$

We have

$$|l_1 - l_2| = |l_1 + f(z) - f(z) - l_2|$$
$$= |f(z) - l_1 + f(z) - l_2|$$
$$\leq |f(z) - l_1| + |f(z) - l_2| \text{ (using Triangle inequality)}$$
$$< \frac{\xi}{2} + \frac{\xi}{2} = \frac{2\xi}{2}$$
$$< \xi$$
$$|l_1 - l_2| < \xi$$

Since ξ is arbitrary,

$$|l_1 - l_2| = 0$$

$$\therefore l_1 = l_2$$

Example:1

Prove That $\lim_{z\to 2} \frac{z^2-4}{z-2} = 4.$

Solution:

Let $f(z) = \frac{z^2-4}{z-2}$, hence f(z) is not defined at $z = 2$ and when $z \neq 2$

We have,

$$f(z) = \frac{(z+2)(z-2)}{z-2}$$
$$= z+2$$

$\therefore$

$$\begin{aligned} |f(z)-4| &= |z+2-4| \\ &= |z-2| \end{aligned}, \text{ when } z \neq 2$$

Now given $\xi > 0$, we choose $\delta = \xi$ then $0 < |z-2| < \delta$

$$\Rightarrow |f(z)-4| < \xi$$

$$\therefore \lim_{z\to 2} f(z) = 4.$$

Example:2

Prove that the function $f(z) = \frac{\bar{z}}{z}$ does not have a limit as $z \to 0$.

PROOF:

Given $f(z) = \frac{\bar{z}}{z}$

$$f(z) = \frac{x - iy}{x + iy}$$

Suppose $z \to 0$ along the path $y=mx$

Along this path

$$f(z)=\frac{x-imx}{x+imx}$$

$$f(z)=\frac{x(1-im)}{x(1+im)} \qquad x\neq 0$$

$$f(z)=\frac{1-im}{1+im}$$

Remark: 1

Let f and g be the functions whose limits at z_0 exist. Let $\underset{z\to z_0}{lt}\ f(z)=l.$ and $\underset{z\to z_0}{lt}\ f(z)=m$. Then

i) $\underset{z\to z_0}{lt}\left(f(z)+g(z)\right)=l+m$

ii) $\underset{z\to z_0}{lt}\ f(z)g(z)=lm$

iii) $\underset{z\to z_0}{lt}\ \dfrac{f(z)}{g(z)}=\dfrac{l}{m}$, $m\neq 0$

Remark:2

i) If $\underset{z \to z_0}{lt} f(z) = l$ then $\underset{z \to z_0}{lt} \overline{f(z)} = \bar{l}$

ii) If $\underset{z \to z_0}{lt} f(z) = l$ then $\underset{z \to z_0}{lt} |f(z)| = |l|$

iii) If $\underset{z \to z_0}{lt} f(z) = l$ iff $\underset{z \to z_0}{lt} \operatorname{Re} f(z) = \operatorname{Re} l$ and $\underset{z \to z_0}{lt} \operatorname{Im} f(z) = \operatorname{Im} l$

Continuous Function :

Let f be a complex valued function defined on a region D then f is said to be continuous at z_0,

If $\underset{z \to z_0}{lt} f(z) = f(z_0)$.Thus , f is continuous at z_0 ,if given $\xi > 0$, there exist a $\delta > 0$ such that

. $0 < |z - z_0| < \delta \Rightarrow |f(z) - f(z_{)0}| < \xi$.

f is said to be continuous in D, if it is continuous at each point of D.


thank you